General Certificate of Education
Advanced Level Examination
January 2013

Mathematics MFP3

Unit Further Pure 3

Friday 25 January 2013 1.30 pm to 3.00 pm

For this paper you must have:
- the blue AQA booklet of formulae and statistical tables.

You may use a graphics calculator.

Time allowed
- 1 hour 30 minutes

Instructions
- Use black ink or black ball-point pen. Pencil should only be used for drawing.
- Fill in the boxes at the top of this page.
- Answer **all** questions.
- Write the question part reference (eg (a), (b)(i) etc) in the left-hand margin.
- You must answer each question in the space provided for that question. If you require extra space, use an AQA supplementary answer book; do **not** use the space provided for a different question.
- Do not write outside the box around each page.
- Show all necessary working; otherwise marks for method may be lost.
- Do all rough work in this book. Cross through any work that you do not want to be marked.

Information
- The marks for questions are shown in brackets.
- The maximum mark for this paper is 75.

Advice
- Unless stated otherwise, you may quote formulae, without proof, from the booklet.
- You do not necessarily need to use all the space provided.

1 It is given that $y(x)$ satisfies the differential equation

$$\frac{dy}{dx} = f(x, y)$$

where $$f(x, y) = \sqrt{2x + y}$$

and $$y(3) = 5$$

(a) Use the Euler formula

$$y_{r+1} = y_r + hf(x_r, y_r)$$

with $h = 0.2$, to obtain an approximation to $y(3.2)$, giving your answer to four decimal places. *(3 marks)*

(b) Use the formula

$$y_{r+1} = y_{r-1} + 2hf(x_r, y_r)$$

with your answer to part **(a)**, to obtain an approximation to $y(3.4)$, giving your answer to three decimal places. *(3 marks)*

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2 (a) Write down the expansion of e^{3x} in ascending powers of x up to and including the term in x^2. *(1 mark)*

(b) Hence, or otherwise, find the term in x^2 in the expansion, in ascending powers of x, of $e^{3x}(1 + 2x)^{-\frac{3}{2}}$. *(4 marks)*

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3 It is given that the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$$

is $y = e^x(Ax + B)$. Hence find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 6e^x$$ *(5 marks)*

4 (a) Explain why $\int_0^1 x^4 \ln x \, \mathrm{d}x$ is an improper integral. *(1 mark)*

(b) Evaluate $\int_0^1 x^4 \ln x \, \mathrm{d}x$, showing the limiting process used. *(6 marks)*

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5 (a) Show that $\tan x$ is an integrating factor for the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\sec^2 x}{\tan x} y = \tan x$$

(2 marks)

(b) Hence solve this differential equation, given that $y = 3$ when $x = \frac{\pi}{4}$. *(6 marks)*

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6 (a) It is given that $y = \ln(\mathrm{e}^{3x} \cos x)$.

(i) Show that $\frac{\mathrm{d}y}{\mathrm{d}x} = 3 - \tan x$. *(3 marks)*

(ii) Find $\frac{\mathrm{d}^4 y}{\mathrm{d}x^4}$. *(3 marks)*

(b) Hence use Maclaurin's theorem to show that the first three non-zero terms in the expansion, in ascending powers of x, of $\ln(\mathrm{e}^{3x} \cos x)$ are $3x - \frac{1}{2}x^2 - \frac{1}{12}x^4$. *(3 marks)*

(c) Write down the expansion of $\ln(1 + px)$, where p is a constant, in ascending powers of x up to and including the term in x^2. *(1 mark)*

(d) (i) Find the value of p for which $\lim_{x \to 0} \left[\frac{1}{x^2} \ln\left(\frac{\mathrm{e}^{3x} \cos x}{1 + px} \right) \right]$ exists.

(ii) Hence find the value of $\lim_{x \to 0} \left[\frac{1}{x^2} \ln\left(\frac{\mathrm{e}^{3x} \cos x}{1 + px} \right) \right]$ when p takes the value found in part **(d)(i)**. *(4 marks)*

7 (a) Find the general solution of the differential equation

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 6\frac{\mathrm{d}y}{\mathrm{d}t} + 10y = \mathrm{e}^{2t}$$

giving your answer in the form $y = \mathrm{f}(t)$. *(6 marks)*

(b) Given that $x = t^{\frac{1}{2}}$, $x > 0$, $t > 0$ and y is a function of x, show that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 4t\frac{\mathrm{d}^2y}{\mathrm{d}t^2} + 2\frac{\mathrm{d}y}{\mathrm{d}t}$$ *(5 marks)*

(c) Hence show that the substitution $x = t^{\frac{1}{2}}$ transforms the differential equation

$$x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - (12x^2 + 1)\frac{\mathrm{d}y}{\mathrm{d}x} + 40x^3y = 4x^3\mathrm{e}^{2x^2}$$

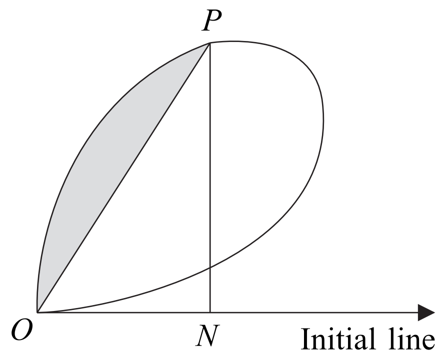
into

$$\frac{\mathrm{d}^2y}{\mathrm{d}t^2} - 6\frac{\mathrm{d}y}{\mathrm{d}t} + 10y = \mathrm{e}^{2t}$$ *(2 marks)*

(d) Hence **write down** the general solution of the differential equation

$$x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - (12x^2 + 1)\frac{\mathrm{d}y}{\mathrm{d}x} + 40x^3y = 4x^3\mathrm{e}^{2x^2}$$ *(1 mark)*

8 The diagram shows a sketch of a curve.



The polar equation of the curve is

$$r = \sin 2\theta \sqrt{\left(2 + \frac{1}{2}\cos\theta\right)}, \quad 0 \leqslant \theta \leqslant \frac{\pi}{2}$$

The point P is the point of the curve at which $\theta = \frac{\pi}{3}$.

The perpendicular from P to the initial line meets the initial line at the point N.

(a) (i) Find the exact value of r when $\theta = \frac{\pi}{3}$. *(2 marks)*

(ii) Show that the polar equation of the line PN is $r = \frac{3\sqrt{3}}{8}\sec\theta$. *(2 marks)*

(iii) Find the area of triangle ONP in the form $\frac{k\sqrt{3}}{128}$, where k is an integer. *(2 marks)*

(b) (i) Using the substitution $u = \sin\theta$, or otherwise, find $\int \sin^n\theta\cos\theta\,\mathrm{d}\theta$, where $n \geqslant 2$.
(2 marks)

(ii) Find the area of the shaded region bounded by the line OP and the arc OP of the curve. Give your answer in the form $a\pi + b\sqrt{3} + c$, where a, b and c are constants.
(8 marks)